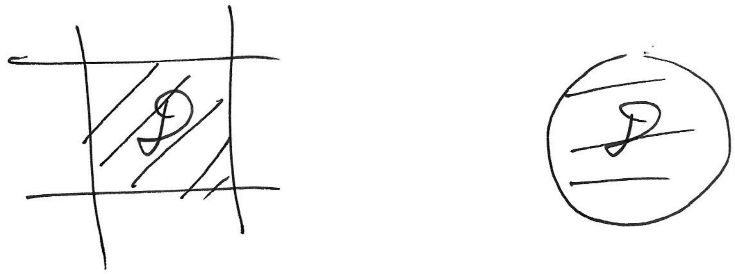
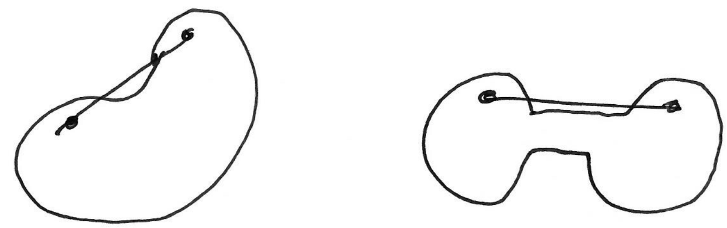
Def. (Convex set)

Convex:



Nonconvex:



$\mathcal{D} \subset \mathbb{R}^2$ is convex if for all (t_1, y_1), (t_2, y_2) in $\mathcal{D}$ The linear segment l between them is in $\mathcal{D}$

or

$$\Big((1-\lambda)t_1 + \lambda t_2,\ (1-\lambda) y_1 + \lambda y_2 \Big) \in \mathcal{D}.$$

for all $\lambda \in [0,1]$.

Thm 5.3 f defined in $\mathcal{D} \subset \mathbb{R}^2$ Convex.

If there is a constant $L>0$ such that

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \le L, \quad \text{for all } (t,y) \in \mathcal{D}.$$

then f satisfies a Lipschitz condition on $\mathcal{D}$ in the variable y with Lipschitz constant L.

<u>Proof:</u>- Consider $g(y) = f(t,y)$ for t fixed and $(t,y) \in \mathcal{D}$.

g is Continuous and differentiable for all $(t,y) \in \mathcal{D}$.
then applying the MVT

$$\frac{|f(t,y_1) - f(t,y_2)|}{|y_1 - y_2|} = \left| \frac{\partial f}{\partial y}(t,\xi) \right|$$

for all $(t,y_1), (t,y_2)$ in $\mathcal{D}$.

Then,

$$|f(t,y_1) - f(t,y_2)| = \left| \frac{\partial f}{\partial y}(t,\xi) \right| |y_1 - y_2|$$
$$\leq L |y_1 - y_2|.$$

So,

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L, \text{ for all } (t,y) \in \mathcal{D}$$

is a Sufficient Condition for a Lipschitz condition.

Thm 5.4

1) $f(t,y)$ conts in $\mathcal{D} = \{(t,y) : a \le t \le b, -\infty < y < \infty\}$

2) f satisfies a Lipschitz condition in $\mathcal{D}$ in the Variable y

then, the IVP:
$$\begin{cases} y'(t) = f(t,y), \quad a \le t \le b & (3.1) \\ y(a) = \alpha & (3.2) \end{cases}$$

has a solution $y(t)$ and this solution is unique in $[a,b]$.

Example: Consider the IVP:

$$y'(t) = \underbrace{4 - t + 2y(t)}_{f(t,y)}, \quad y(0) = 1$$

$$|f(t,y_1) - f(t,y_2)| = 2|y_1 - y_2| \quad \text{for all } t \in \mathbb{R} \text{ and } y \in \mathbb{R}$$

Also, $$\frac{\partial f}{\partial y}(t,y) = 2$$

$\Rightarrow \boxed{L = 2}$ and IVP (3.1)-(3.2) has a unique soln for $t \in \mathbb{R}$, because f is also conts. in $\mathbb{R}^2$.

Well - Posed Problems.

Def The IVP:

$$y' = f(t,y), \quad y(a) = \alpha, \quad a \le t \le b.$$

is well-posed if

i) It has a unique soln.

ii) For a small error in the defining equation; as well as an error in the initial condition, there is still a unique soln. somehow close to the soln. of the original problem without any error (actual soln.).

In math terms:

the IVP:
$$\begin{cases} \dfrac{dz}{dt} = f(t,z) + \delta(t) & a \le t \le b \quad (4.1) \\ z(a) = a + \delta_0 & \quad (4.2) \end{cases}$$

where $|\delta(t)| < \varepsilon$, for all t and $|\delta_0| < \varepsilon$

Then, there is a unique soln. for (4.1)-(4.2) and there is $K > 0$ such that

$$|z(t) - y(t)| < K\varepsilon, \quad \text{for all } t \in [a,b].$$

#6) a) $y' = 1 - y, \quad y(0) = 0, \quad 0 \le t \le 2$

<u>Exact soln:</u> $y' + y = 1$

$y'e^{t} + ye^{t} = e^{t} \Rightarrow (ye^{t})' = e^{t}$

$ye^{t} = \int e^{t}dt = e^{t} + c \Rightarrow \boxed{y(t) = 1 + ce^{-t}}$

Since $y(0) = 0 \Rightarrow 0 = 1 + C \Rightarrow \boxed{C = -1}$

and $\boxed{y(t) = -e^{-t} + 1}$ Exact soln.

Perturbed problem in the next page.

Perturbed Problem: (Sect. 5.1) (# 6 a Textbook)

IVP: $Z' = 1 - z + \delta t$, $\quad z(0) = 0 + \delta_0 = \delta_0, \quad 0 \le t \le 2.$

$\delta(t) = \delta t < \varepsilon$, for all t
$\delta_0 < \varepsilon.$

error grows linearly with t.

$$z' + z = 1 + \delta t \Rightarrow \left(z e^t\right)' = (1+\delta t)\, e^t$$

$$\Rightarrow z e^t = e^t + \delta(t e^t - e^t) + c$$

$$\Rightarrow z(t) = 1 + \delta(t-1) + c e^{-t}$$

I.C: $\delta_0 = z(0) = 1 - \delta + c \Rightarrow c = \delta - 1 + \delta_0$

$$\Rightarrow z(t) = 1 + \delta(t-1) + (\delta - 1 + \delta_0)\, e^{-t}$$

or $\boxed{z(t) = 1 - e^{-t} + \delta\left(t - 1 + e^{-t}\right) + \delta_0 e^{-t}}$

Then,

$$|y(t) - z(t)| \le |\delta|\, |\overbrace{t - 1 + e^{-t}}^{=g(t)}| + |\delta_0|\, e^{-t}$$

$g(0) = -1 + 1 = 0$
$g(2) = 1 + e^{-2}$
$g'(t) = 1 - e^{-t} > 0$ for all $t > 0$

$$\le |\underbrace{1 + e^{-2}}_{\text{at } t=2 \text{ max}}|\, |\delta| + |\delta_0|$$

$$\le 2|\delta| + |\delta_0| < 2\varepsilon$$

the problem is well-posed in $[0,2]$.

Rmk: Notice $\delta t < \varepsilon$ for all t and $0 \le t \le 2 \Rightarrow 2\delta < \varepsilon \Rightarrow \delta < \varepsilon/2$
Then for ε arbitrary, $\delta < \varepsilon/2$ and $\delta_0 < \varepsilon \Rightarrow K = 2$. for well-posedness.